HAMILTONIAN CYCLE PUZZLES

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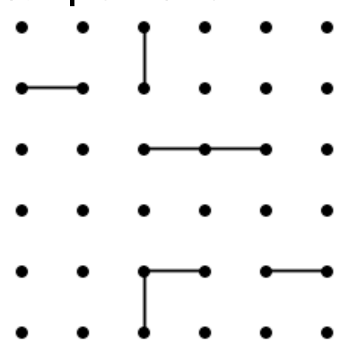
Introduction

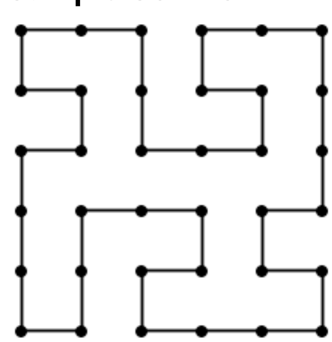
In the mathematics of graph theory, graphs are structures consisting of **vertices** and **edges**, where each edge connects two vertices. A **path** is a sequence of edges in which consecutive edges share a common vertex. A **cycle** is a path that starts and ends at the same vertex. A **Hamiltonian cycle** (or path) is one which includes every vertex exactly once in the cycle (or path). This contrasts with Eulerian cycles (paths) which must traverse every edge of the graph exactly once (but may visit vertices multiple times).

These ideas can be used to construct interesting puzzles, based on the graph induced on a regular grid. Consider a rectangular array of vertices, equally spaced in rows and columns. The edges of the graph are pairs of vertices that neighbor one another either vertically or horizontally. In general there may be many Hamiltonian cycles in such a grid, but by specifying that certain edges **must** be used in the solution path, it can be arranged for the solution cycle to be unique. Examples of such puzzles have been published under names such as **RoundTrip** and **GrandTour**. Here is a sample problem and its solution:

Sample Problem:



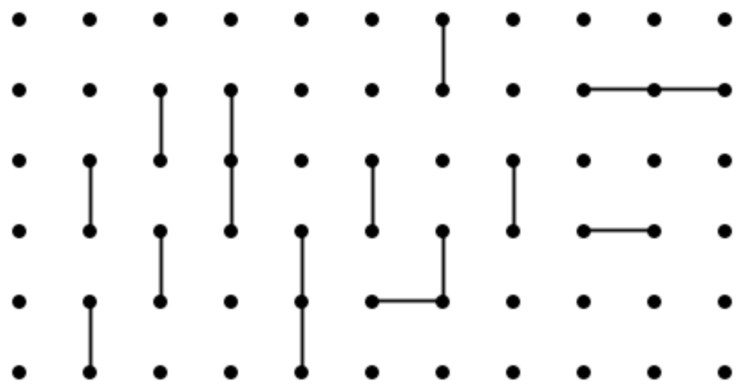
Sample Solution:



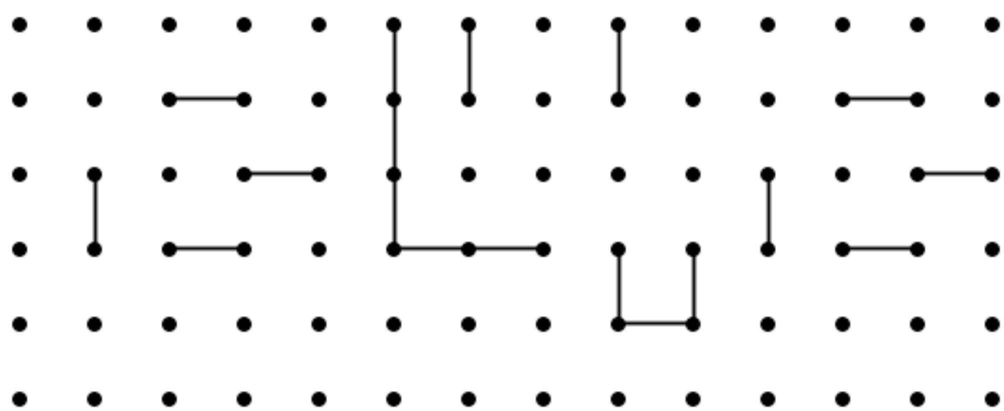
Instructions

In each of the following puzzles, find the unique Hamiltonian cycle that includes all of the edges drawn explicitly in the problem diagram. Once you have found the solution path, shade in the interior space enclosed by the cycle, and you should discover a relevant design.

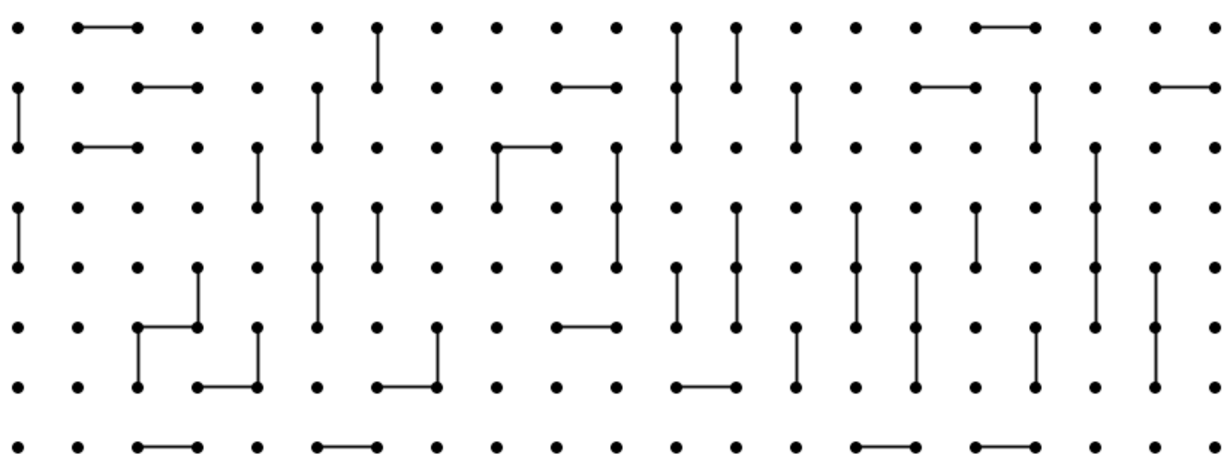
Warm-up puzzle in honor of Martin Gardner:



The Gathering:



G4G8 Theme Puzzle:



Hints and strategies for solving

1. Don't guess. There is only 1 solution, so guessing is unlikely to succeed.

2. Note that the final path must enter and exit each vertex exactly once, so each vertex will end up with exactly two edges connected to it in the final cycle. At any stage of the solving process, each vertex will have 0, 1, or 2 edges connected to it. Once a vertex has 2 edges, it is "used up" or **filled**. It is no longer available for connections. A vertex with 0 edges can be called **open**, and a vertex with exactly 1 edge we term an **endpoint** (since it is situated at the end of a partial path). We say that both open vertices and endpoints are **available** (to be connected to).

3. Try to find edges that must necessarily be part of the solution path. If some open vertex has exactly 2 available neighbors, then it must connect to both of these neighbors. The 4 corner vertices are examples of this, but you may find other occurrences in the initial problem grid or later on as solving progresses. Similarly for an endpoint, if it has exactly 1 available neighbor, then it must connect to that neighbor.

4. If an edge would connect the two endpoints of a single partial path, creating a cycle having fewer than all the vertices, then that edge can be ruled out. It can't be part of the solution path. Having ruled out that edge, there may be only 1 valid alternative for extending one (or both) of the endpoints of that partial path.

5. If some edge placement results in the isolation of another vertex (i.e. leaves an open vertex with 0 or 1 available neighbors, or leaves a half-open vertex with no available neighbors) then that edge can be ruled out. Similarly, if a hypothetical choice leads to a forced isolation, then that hypothetical choice can be ruled out.

6. If an open vertex has exactly 3 available neighbors, then exactly 2 of the 3 possible edges must appear in the final path. If there are 2 of these 3 edges that together would create a cycle (or can be ruled out in some other way), then you can conclude that the 3rd edge is necessarily part of the desired solution path.

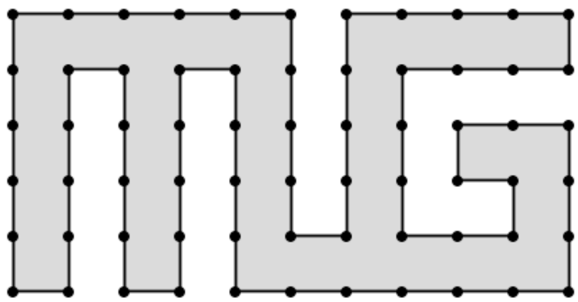
7. As more and more vertices become closed, they may begin to divide the grid into regions with loose boundaries. There may be partial paths that enter such regions. Note that any solution path must cross any region boundary an even number of times (since each entry to the region must have a corresponding exit as the path is traversed). If adding some edge creates a situation where only an odd number of partial paths could cross the boundary of a region, then that edge can be ruled out.

If you like these kinds of problems

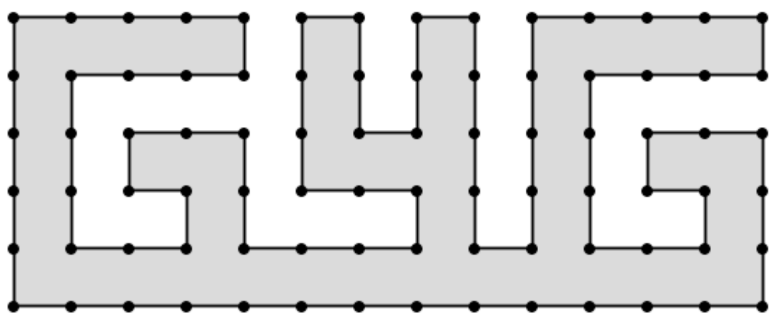
I invite you to visit my web site: http://glenniba.com/
There you can try out many more of my GrandTour puzzles on-line. There are also hexagonal variations using triangular grids.

Puzzle Solutions

Warm-up puzzle in honor of Martin Gardner:



The Gathering:



G4G8 Theme Puzzle: